

Finiteness of certain groups attached to algebraic groups over a finite field

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September 21, 2012

Abstract

Let G_1, \dots, G_q be algebraic varieties over a finite field k . We show that, if $q \geq 2$, the finiteness of the tensor product of G_1, \dots, G_q as Mackey functors. We apply this to prove the finiteness of a relative Chow group and an abelian fundamental group which classifies abelian coverings with bounded ramification along the boundary.

1 Introduction

Algebraic groups G_1, \dots, G_q over a field k are regarded as Mackey functors in the sense of [4] and their product $G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k)$ is defined by

$$(1) \quad \left(\bigoplus_{E/k: \text{finite}} G_1(E) \otimes \cdots \otimes G_q(E) \right) / R,$$

where R is the subgroup generated by elements of the following form:

(PF) For any finite field extensions $k \subset E_1 \subset E_2$, and if $x_{i_0} \in G_{i_0}(E_2)$ and $x_i \in G_i(E_1)$ for $i \neq i_0$, then

$$j^*(x_1) \otimes \cdots \otimes x_{i_0} \otimes \cdots \otimes j^*(x_q) - x_1 \otimes \cdots \otimes j_*(x_{i_0}) \otimes \cdots \otimes x_q,$$

where $j : \text{Spec}(E_2) \rightarrow \text{Spec}(E_1)$ is the canonical map and j^* and j_* are the pull-back and push-forward along j respectively.

The product $G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k)$ is called the *Mackey product* of G_1, \dots, G_q and it gives a tensor structure in the abelian category of Mackey functors.

Recently, F. Ivorra and K. Rülling [3] introduced the Milnor type K -group $K(k; G_1, \dots, G_q)$ associated to the algebraic groups G_1, \dots, G_q as a quotient of the Mackey product $G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k)$ under assumptions that the field k is finitely generated over a perfect field and the characteristic of k is $\neq 2$ (for the precise definition, see [3] Def. 4.2.3). In fact, this group is an extension of Somekawa's K -group [12] which was limited on considering only semi-abelian varieties. For semi-abelian varieties G_1, \dots, G_q in [5], B. Kahn showed that

$$(2) \quad K(k; G_1, \dots, G_q) = G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k) = 0$$

when k is a finite field and $q > 1$. This work generalizes the classical fact that the Milnor K -group $K_q^M(k)$ of the finite field k becomes trivial for $q > 1$, because of the isomorphism ([12], Thm. 1.4)

$$K(k; \overbrace{\mathbb{G}_m, \dots, \mathbb{G}_m}^q) \xrightarrow{\simeq} K_q^M(k).$$

For general algebraic groups here we prove the following theorem on both of the Mackey product and the Somekawa K -group.

Theorem 1.1 (Thm. 2.1, 2.2). *Let G_1, \dots, G_q be algebraic groups over a finite field k with characteristic $\neq 2$ for $q > 1$. Then we have*

$$K(k; G_1, \dots, G_q) = 0, \quad G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_q(k) \neq 0 \text{ and is finite.}$$

As an application of (2), the class field theory of the product of projective smooth curves over a finite field (a special case of higher dimensional class field theory of S. Bloch, K. Kato and S. Saito [7]) is reduced from the classical (unramified) class field theory (= class field theory of curves over a finite field) and Lang's theorem; the reciprocity map on a normal variety over a finite field has dense image. We will pursue related results on the (ramified) class field theory for the product of open curves as a byproduct of the above main theorem.

Acknowledgments. A part of this note was written during a stay of the author at the Duisburg-Essen university. He thanks the institute for its hospitality. Most of what the author know about relative Chow groups and Albanese varieties from Henrik Russell. This work was supported by KAKENHI 21740015.

2 Finiteness

First we recall some properties of the Mackey product and the Milnor type K -group (here we just call it the Somekawa K -group) attached to algebraic groups G_1, \dots, G_q over a field k . Here we mean by an *algebraic group* a smooth connected and commutative group scheme over a field.

On the Mackey product defined in (1), we have

- (M1) The Mackey product gives a tensor product in the abelian category of the Mackey functors with unit $\mathbb{Z} : E \mapsto \mathbb{Z}$ which is given by identity maps on \mathbb{Z} .
- (M2) The product $- \otimes^M G$ is right exact for any algebraic group G .
- (M3) For any finite field extension k'/k , putting $G_1 \otimes^M \dots \otimes^M G_q(k') := (G_1 \otimes k') \otimes^M \dots \otimes^M (G_q \otimes k')(k')$ the push-forward $N_{k'/k} : G_1 \otimes^M \dots \otimes^M G_q(k') \rightarrow G_1 \otimes^M \dots \otimes^M G_q(k)$ is given by $N_{k'/k}(\{x_1, \dots, x_q\}_{E'/k'}) = \{x_1, \dots, x_q\}_{E'/k}$ on symbols. Here we write $\{x_1, \dots, x_q\}_{E/k}$ for the image of $x_1 \otimes \dots \otimes x_q \in G_1(E) \otimes \dots \otimes G_q(E)$ in the product $G_1 \otimes^M \dots \otimes^M G_q(k)$.
- (M4) Let U be a unipotent algebraic group and G a semi-abelian variety over k . If k is a perfect field of characteristic > 0 , we have $U \otimes^M G(k) = 0$ ([2], Lem. 3.5)¹.

After Kahn-Yamazaki [6], the Somekawa K -group also should be considered as a “tensor product” in a category of Mackey functors which satisfies some reciprocity condition; reciprocity functors in the sense of [3]. At present we have

- (S1) $K(k; G_1, \dots, G_i \oplus G'_i, \dots, G_q) \simeq K(k; G_1, \dots, G_i, \dots, G_q) \oplus K(k; G_1, \dots, G'_i, \dots, G_q)$.
- (S2) The Somekawa K -group $K(k; G, -)$ is right exact ([3], Cor. 4.2.5).

¹ In [2] we showed only $W \otimes^M G(k) = 0$ for some Witt type group W and a semi-abelian variety G over k . However, by the same arguments as in the proof of Theorem 5.5.1 in [3] (see also the proof of Thm. 2.2 below) we can easily reduce to showing $\mathbb{G}_a \otimes^M G(k) = 0$.

(S3) Assume that $\text{char}(k) \neq 2$. For unipotent groups U_1, \dots, U_q , we have $K(k; U_1, \dots, U_q) = 0$ ([3], Thm. 5.5.1).

First we show the following theorem:

Theorem 2.1. *Let G_1, \dots, G_q be algebraic groups over a finite field k with $\text{char}(k) \neq 2$. Then we have $K(k; G_1, \dots, G_q) = 0$.*

Proof. Here we show that $K(k; G_1, G_2)$ is trivial². The algebraic group G_i has a decomposition $0 \rightarrow T_i \oplus U_i \rightarrow G_i \rightarrow A_i \rightarrow 0$ where T_i is a torus, U_i a unipotent algebraic group and A_i an abelian variety. From the right exactness (M3) we have the following exact sequences

$$(3) \quad \begin{array}{ccccccc} K(k; T_1 \oplus U_1, T_2 \oplus U_2) & & & & K(k; A_1, T_2 \oplus U_2) & & \\ \downarrow & & & & \downarrow & & \\ K(k; T_1 \oplus U_1, G_2) & \longrightarrow & K(k; G_1, G_2) & \longrightarrow & K(k; A_1, G_2) & \longrightarrow & 0 \\ \downarrow & & & & \downarrow & & \\ K(k; T_1 \oplus U_1, A_2) & & & & K(k; A_1, A_2) & & \\ \downarrow & & & & \downarrow & & \\ 0 & & & & 0 & & \end{array} .$$

From (M4)

$$K(k; T_1 \oplus U_1, T_2 \oplus U_2) \simeq K(k; T_1, T_2) \oplus K(k; T_1, U_2) \oplus K(k; U_1, T_2) \oplus K(k; U_1, U_2)$$

and $K(k; T_1 \oplus U_1, A_2) \simeq K(k; T_1, A_2) \oplus K(k; U_1, A_2)$. We obtain

$$K(k; A_1, A_2) = K(k; T_1, T_2) = 0 \quad \text{by Kahn's theorem (2),}$$

$$K(k; T_1, U_2) = K(k; T_2, U_1) = 0 \quad \text{by (M4),}$$

$$K(k; U_1, U_2) = 0 \quad \text{by (S3).}$$

The same arguments work on the term $K(k; A_1, T_2 \oplus U_2)$. Therefore, we have $K(k; G_1, G_2) = 0$. \square

² The proof is exactly same for the case of $q > 2$ by using the right exactness of (M2) according to the decomposition of algebraic groups as in (3). Note also that the Somekawa K -group does *not* satisfy the associativity like $K(k; G_1, G_2, G_3) \simeq K(k; K(k; G_1, G_2), G_3)$ considering in the category of reciprocity functors ([3], Rem. 4.2.6). Hence we cannot reduce the assertion to $q = 2$.

Next we consider the Mackey product of algebraic groups over a finite field which may be assumed $\text{char}(k) = 2$.

Theorem 2.2. *Let G_1, \dots, G_q be algebraic groups over a finite field k . Then we have $G_1 \overset{M}{\otimes} \dots \overset{M}{\otimes} G_q(k)$ is finite and non-zero.*

Proof. We may assume that $q = 2$. The algebraic group G_i has a decomposition $0 \rightarrow T_i \oplus U_i \rightarrow G_i \rightarrow A_i \rightarrow 0$ as above and replacing the Somekawa K -group by the Mackey product in (3) it is enough to show the assertion for $G_1 = U_1$ and $G_2 = U_2$. The unipotent group $G := G_i$ has a composition series:

$$0 = G^r \subset \dots \subset G^1 \subset G,$$

each G^i/G^{i+1} being isomorphic to \mathbb{G}_a ([3], Proof of Thm. 5.5.1). By the right exactness (M2), we may assume $G_1 = G_2 = \mathbb{G}_a$ without loss of generality. The “non-zero” part is easy to show by constructing a well-defined surjection

$$\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k) \rightarrow k; \quad \{x, y\}_{E/k} \mapsto \text{Tr}_{E/k}(xy).$$

The group $\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k)$ has a structure of a k -vector space given by $a\{x, y\}_{E/k} := \{ax, y\}_{E/k}$ for any $a \in k$ and a symbol $\{x, y\}_{E/k}$. Consider a subspace $I(k)$ of $\mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k)$ generated by the elements of the form $\{x, 1\}_{E/k} - \{1, x\}_{E/k}$. Since the push-forward map on \mathbb{G}_a is just the trace map and we have $\{x, 1\}_{E/k} = \{\text{Tr}_{E/k}(x), 1\}_{k/k}$, the subspace $I(k)$ is finite. Next we consider the subspace $S(k)$ of the quotient $Q(k) := \mathbb{G}_a \overset{M}{\otimes} \mathbb{G}_a(k)/I(k)$ generated by symbols of the form $\overline{\{x, y\}_{k/k}}$. Here we denote by $\overline{\{x, y\}_{E/k}}$ the image of $\{x, y\}_{E/k}$ in the quotient $Q(k)$. It is easy to see that the subspace $S(k)$ is finite. (In fact, $S(k) \xrightarrow{\cong} k$ given by $\overline{\{x, y\}_{k/k}} \mapsto xy$.) For any symbol $\overline{\{x, y\}_{E/k}}$ in $Q(k)$, we have

$$\begin{aligned} \overline{\{x, y\}_{E/k}} &= N_{E/k}(\overline{\{x, y\}_{E/E}}) \quad \text{by (M3)} \\ &= N_{E/k}(\overline{\{xy, 1\}_{E/E}}) \quad \text{because of } \overline{\{x, y\}_{E/E}} \in S(E) \\ &= \overline{\{xy, 1\}_{E/k}} \quad \text{by (M3)} \\ &= \{\text{Tr}_{E/k}(xy), 1\}_{k/k} \quad \text{by (PF)}. \end{aligned}$$

Thus we obtain $Q(k) = S(k)$ and the assertion follows from it. \square

3 Applications

Let X be a projective smooth variety over a finite field k , D an effective Weil divisor on X . To consider the ramification along the divisor on D here we recall the relative Chow group $\mathrm{CH}_0(X, D)$ of the pair (X, D) after H. Russell ([9], Sect. 3.4 and 3.5). We denote by $R_0(X, D)$ a set of pairs (C, f) of C a curve in X intersecting D properly and $f \in k(C)^\times$ such that $f \equiv 1 \pmod{(D - D_{\mathrm{red}}) \cdot \tilde{C} + (D \cdot \tilde{C})_{\mathrm{red}}}$, where $\tilde{C} \rightarrow C$ is the normalization and $D \cdot \tilde{C}$ denotes the pull-back of D to \tilde{C} . Now we define

$$\mathrm{CH}_0(X, D) := \mathrm{Coker}(\mathrm{div} : R_0(X, D) \rightarrow Z_0(X \setminus D)),$$

where the map div is given by the divisor map on each curve C as $(C, f) \mapsto \mathrm{div}_C(f)$. Putting $X_E := X \otimes E$, the assignment

$$\mathcal{CH}_0(X, D) : E \mapsto \mathrm{CH}_0(X_E, D_E)$$

gives a Mackey functor $\mathcal{CH}_0(X, D)$. As an application of the main theorem, we show the following finiteness theorem of the relative Chow group:

Theorem 3.1. *Let X_1, \dots, X_q be projective smooth and geometrically irreducible curves over a finite field k , and D_i an effective Weil divisor on X_i . Put $X := X_1 \times \dots \times X_q$ and $D := p_1^* D_1 + \dots + p_q^* D_q$ where $p_i : X = X_1 \times \dots \times X_q \rightarrow X_i$ is the projection. Assume that $(X_i \setminus D_i)(k) \neq \emptyset$. Then, the kernel of the degree map $\mathrm{CH}_0(X, D)^0 := \mathrm{Ker}(\mathrm{deg} : \mathrm{CH}_0(X, D) \rightarrow \mathbb{Z})$ is finite.*

Remark. H. Esnault and M. Kerz recently showed the finiteness of the group $\mathrm{CH}_0(X, D)^0$ for more general variety over a finite field as an application of Deligne's finiteness theorem for l -adic Galois representations of function fields ([1], Thm. 2.5).

Proof of Thm. 2.2. It is enough to show the case $q = 2$. Consider the map

$$(4) \quad \psi : \mathcal{CH}_0(X_1, D_1) \overset{M}{\otimes} \mathcal{CH}_0(X_2, D_2)(k) \rightarrow \mathrm{CH}_0(X, D)$$

defined by $\{x_1, x_2\}_{E/k} \mapsto (j_{E/k})_*(p_1^* x_1 \cap p_2^* x_2)$, where $X = X_1 \times X_2$, $D = D_1 \times X_2 + X_1 \times D_2$, $j_{E/k} : X_E \rightarrow X$ is given by the base change to E and \cap is the internal product. We show that the map ψ is surjective. Take a closed point P on $X \setminus D$ as a generator of $\mathrm{CH}_0(X, D)$. By the definition of ψ ,

the push-forward map on the relative Chow group and the norm map on the Mackey product are compatible as in the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{CH}_0(X_1, D_1) \otimes^M \mathcal{CH}_0(X_2, D_2)(k(P)) & \xrightarrow{\psi_{k(P)}} & \mathrm{CH}_0(X_{k(P)}, D_{k(P)}) \\
\downarrow N_{k(P)/k} & & \downarrow (j_{k(P)/k})^* \\
\mathcal{CH}_0(X_1, D_1) \otimes^M \mathcal{CH}_0(X_2, D_2)(k) & \xrightarrow{\psi} & \mathrm{CH}_0(X, D).
\end{array}$$

Thus to show the surjectivity of ψ we may assume that P is a k -rational point. The point P is determined by the closed points P_1 on X_1 and P_2 on X_2 . These points give $\psi(\{P_1, P_2\}_{k/k}) = P$.

From the assumption $(X_i \setminus D_i)(k) \neq \emptyset$, we have the canonical decomposition $\mathrm{CH}_0(X_i, D_i) = \mathbb{Z} \oplus J(X_i, D_i)(k)$ by the generalized Jacobian variety $J(X_i, D_i)$ of the pair (X_i, D_i) ([11]). According to this decomposition we obtain

$$\begin{aligned}
\mathcal{CH}_0(X_1, D_1) \otimes^M \mathcal{CH}_0(X_2, D_2)(k) = \\
\mathbb{Z} \oplus J(X_1, D_1)(k) \oplus J(X_2, D_2)(k) \oplus (J(X_1, D_1) \otimes^M J(X_2, D_2)(k))
\end{aligned}$$

by (M1). From the product (4), there exists a surjection

$$J(X_1, D_1)(k) \oplus J(X_2, D_2)(k) \oplus (J(X_1, D_1) \otimes^M J(X_2, D_2)(k)) \twoheadrightarrow \mathrm{CH}_0(X, D)^0.$$

The left is finite by Theorem 2.2 and so is $\mathrm{CH}_0(X, D)^0$. \square

In terms of the abelian fundamental group we obtain the finiteness of some abelian fundamental group corresponding to abelian coverings with bounded ramification. Let (X_i, D_i) and (X, D) be as in the above theorem. For each curve C on X intersecting properly to D and a point P in $D \cdot \tilde{C}$, we denote by m_P the multiplication of the divisor $(D - D_{\mathrm{red}}) \cdot \tilde{C} + (D \cdot \tilde{C})_{\mathrm{red}}$ at P . For each such C and P we have the canonical map $G_P^{\mathrm{ab}} := \mathrm{Gal}(k(C)_P^{\mathrm{ab}}/k(C)_P) \rightarrow \pi_1(X \setminus D)^{\mathrm{ab}}$, where $k(C)_P^{\mathrm{ab}}$ is the maximal abelian extension of the completion $k(C)_P$ at P . Now we define

$$\pi_1(X, D)^{\mathrm{ab}} := \mathrm{Coker} \left(\bigoplus_{C \subset X} \bigoplus_{P \in \tilde{C} \cdot D} G_P^{\mathrm{ab}, m_P} \rightarrow \pi_1(X \setminus D)^{\mathrm{ab}} \right),$$

where $G_P^{\text{ab},m}$ is the ramification subgroup of G_P^{ab} ([10]) and C runs through the set of curves on X intersecting to D properly. By local class field theory, we have the following commutative diagram:

$$\begin{array}{ccc} Z_0(X \setminus D)^0 & \longrightarrow & \text{CH}_0(X, D)^0 \\ \rho \downarrow & & \downarrow \rho_D \\ \pi_1(X \setminus D)^{\text{ab},0} & \longrightarrow & \pi_1(X, D)^{\text{ab},0}. \end{array}$$

Here the left vertical map is the reciprocity map on $X \setminus D$ and $\pi_1(X, D)^{\text{ab},0}$ is the geometric part of the fundamental group (= the kernel of $\pi_1(X, D)^{\text{ab}} \rightarrow \pi_1(\text{Spec}(k))^{\text{ab}}$). The image of the left vertical map ρ is known to be dense in $\pi_1(X \setminus D)^{\text{ab}}$ (due to Lang, [8]) and the image of ρ_D is finite by Theorem 3.1. Therefore, the map ρ_D is surjective and we obtain the following finiteness result.

Corollary 3.2. $\pi_1(X, D)^{\text{ab},0}$ is finite.

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